GAUSSIAN ESTIMATES FOR A HEAT EQUATION ON A NETWORK

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Abstract. We consider a diffusion problem on a network on whose nodes we impose Dirichlet and generalized Kirchhoff-type conditions. We prove well-posedness of the associated initial value problem, and we exploit the theory of sub-Markovian and ultracontractive semigroups in order to obtain upper Gaussian estimates for the heat kernel. We conclude that the same diffusion problem is governed by an analytic semigroup acting on all L^p-spaces as well as on a suitable space of continuous functions. Stability and spectral issues are also discussed.

1. Introduction

After the pioneering works of, among others, von Below ([4], [5]) and Nicaise ([16]), in the last decade evolution equations on networks have aroused broad interest again. Among those papers dealing with this kind of problems by means of operator, graph theoretical, and/or stochastical methods, we mention [6] (heat equations), [11] and [14] (flows), [10] (general diffusion and wave equations), and [18] (Schrödinger equations).

In this paper we pursue an approach to parabolic equations on networks based on the theory of sesquilinear forms and associated sub-markovian semigroups. In particular, as an application of the abstract results presented, e.g., in [7] and [17], we can obtain Gaussian estimates for the semigroups that govern such problems.

Throughout this paper we consider a finite, unitarily parametrized, connected network whose structure is given by a suitable graph. On it we study a general diffusion equation. Adopting a setting which is standard in literature, the node conditions impose continuity and Kirchhoff laws in ramification nodes.

Kramar, Sikolya, and the author have studied in [10] diffusion equations on networks on which Kirchhoff laws are imposed on each node (none of which is a boundary one – i.e., all the vertices of the underlying graph have degree ≥ 2). What happens if boundary nodes are instead considered? If a vertex has degree 1, i.e., it is the endpoint of only one edge, then physical and mechanical considerations motivate to impose on it a Dirichlet condition, cf. [12, Chapt. 2].

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Consider thus a network whose underlying graph has n_0 vertices of degree 1, i.e., a network where n_0 edges have an endpoint that is a boundary node, on which Dirichlet conditions are in fact imposed. Since no process takes place in such boundary vertices, we may and do identify all of them. Summing up, we instead consider an equivalent diffusion equation on a rearranged graph where these n_0 nodes of degree 1 are replaced by *only one node of degree* n_0, on which a Dirichlet condition is imposed.

This case is not really technically harder to treat than that of purely Kirchhoff node conditions discussed in [10]. However, in this setting we are able to draw interesting conclusions about several issues, including L^∞-contractivity of the semigroup governing the problem, its L^2-L^∞-stability and L^1-analyticity, and – most important – well-posedness of the initial value diffusion problem in a suitable space of continuous functions. The key arguments are the invertibility of the operator associated with the problem and sharp upper bound estimates for the heat kernel of the generated semigroup. Observe that while the heat kernel for such a problem has already been explicitly obtained in [16] (see also [6]) in the special case of constant coefficients for the heat operator, Gaussian estimates for the heat kernel are, to our knowledge, completely new in this field in the case of *variable* coeffients.

2. General framework

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1,\dots,\mathsf{e}_m$ and n vertices $\mathsf{v}_1,\dots,\mathsf{v}_n$. We normalize and parametrize the edges on the interval $[0,1]$. The structure of the network is given by the $n\times m$ matrices $\Phi^+ := (\phi^+_{ij})$ and $\Phi^- := (\phi^-_{ij})$ defined by

$$\phi^+_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(0)=\mathsf{v}_i \text{ and } i\geq 2,\\ 0, & \text{otherwise,}\end{cases} \qquad \text{and} \qquad \phi^-_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(1)=\mathsf{v}_i \text{ and } i\geq 2,\\ 0, & \text{otherwise.}\end{cases}$$

The $n\times m$ matrix $\Phi := (\phi_{ij})$, defined by $\Phi := \Phi^+ - \Phi^-$, is thus a (modified[1]) *incidence matrix* of the graph G Further, let $\Gamma(\mathsf{v}_i)$ be the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \{j\in\{1,\dots,m\} : \mathsf{e}_j(0)=\mathsf{v}_i \text{ or } \mathsf{e}_j(1)=\mathsf{v}_i\}, \qquad 1\leq i\leq n.$$

For the sake of simplicity, we denote the value of the functions $c_j(\cdot)$ and $u_j(t,\cdot)$ at 0 or 1 by $c_j(\mathsf{v}_i)$ and $u_j(t,\mathsf{v}_i)$, if $\mathsf{e}_j(0)=\mathsf{v}_i$ or $\mathsf{e}_j(1)=\mathsf{v}_i$, respectively. With an abuse of notation, we also set $u'_j(\mathsf{v}_i)=c_j(\mathsf{v}_i):=0$ whenever $j\notin\Gamma(\mathsf{v}_i)$.

We now consider the diffusion problem

(NDP)
$$\begin{cases} \dot u_j(t,x) &= (c_j u'_j)'(t,x), & t\geq 0,\ x\in(0,1),\ j=1,\dots,m,\\ u_j(t,\mathsf{v}_1) &= 0, & t\geq 0,\ j\in\Gamma(\mathsf{v}_1),\\ u_j(t,\mathsf{v}_i) &= u_\ell(t,\mathsf{v}_i), & t\geq 0,\ j,\ell\in\Gamma(\mathsf{v}_i),\ i=2,\dots,n,\\ \sum_{j=1}^m \phi_{ij}c_j(\mathsf{v}_i)u'_j(t,\mathsf{v}_i) &= b_i u(t,\mathsf{v}_i), & t\geq 0,\ i=2,\dots,n,\\ u_j(0,x) &= u_{0j}(x), & x\in(0,1),\ j=1,\dots,m,\end{cases}$$

[1] In the usual definition of an incidence matrix of a graph the additional condition "$i\geq 2$" is not present. Such a modification is in fact useful to treat the Dirichlet boundary condition we want to impose in v_1.

on the network. The functions $c_1, \ldots, c_m$ represent the different speeds of propagations along each edge of the network G, and throughout this paper we assume that $0 < c_j \in C^1[0,1]$, $j = 1, \ldots, m$, while the constants $b_i \leq 0$, $i = 2, \ldots, n$. The second equation above prescribes a Dirichlet condition in v_1, while the fourth one is a generalized Kirchhoff-type law motivated by applications (see [18]).

Remark 2.1. The second and third equations in (NDP) impose the continuity of the values attained by the system at the nodes, so that for a given v_i it makes sense to denote by $u(t, v_i)$ the joint value of all $u_j(t, v_i)$, $j \in \Gamma(\mathsf{v}_i)$, $t \geq 0$, as we have done in the fourth equation.

We introduce the $n \times m$ matrices $\Phi_w^+ := (\omega_{ij}^+)$ and $\Phi_w^- := (\omega_{ij}^-)$ defined by

$$\omega_{ij}^+ := \begin{cases} c_j(\mathsf{v}_i), & \text{if } \phi_{ij}^+ = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \omega_{ij}^- := \begin{cases} c_j(\mathsf{v}_i), & \text{if } \phi_{ij}^- = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following we will repeatedly and without further notice write the functions u_j in vector form, i.e.,

$$u \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

With these notations, the second, third and fourth equations in (NDP) can be rewritten as

$$\exists d_t \in \mathbb{C}^n \quad \text{s.t.} \quad \begin{aligned} (\Phi^+)^\top d_t = u(t,0),\ (\Phi^-)^\top d_t = u(t,1), \quad \text{and} \\ \Phi_w^+ u'(t,0) - \Phi_w^- u(t,1) = B d_t \quad \text{for all } t \geq 0, \end{aligned} \tag{2.1}$$

where (for $b_1 := 0$) we have introduced the diagonal $n \times n$ matrices $B := (b_i \delta_{ih})$ (here δ_{ih} stands for the Kronecker delta).

Remark 2.2. Observe that for all $t \geq 0$ the vector d_t that appears in (2.1) is in fact given, with the notation introduced in Remark 2.1, by

$$d_t = \begin{pmatrix} 0 \\ u(t, \mathsf{v}_2) \\ \vdots \\ u(t, \mathsf{v}_n) \end{pmatrix}, \quad t \geq 0.$$

As already remarked, at least in the case of $c_1 = \ldots = a_m \equiv 1$ and $b_2 = \ldots = b_n = 0$ the well-posedness of (NDP) has already been shown in a Hilbert space context by von Below (see [4]). Our first goal is to to establish a meaningful L^p-theory. To this aim, we need the following.

Definition 2.3. *For given functions $f_j : [0,1] \to \mathbb{C}$, $j = 1, \ldots, n$, we define a mapping $Uf : [0,1] \to \mathbb{C}^n$ by*

$$Uf(x) := \tilde{f}(x) := f_j(x - j + 1) \qquad \text{if } x \in (j-1, j),\ j = 1, \ldots, m.$$

Lemma 2.4. *The mapping U is one-to-one from $X_p := (L^p(0,1))^m$ onto $L^p(0,m)$ for all $p \in [1, \infty]$, and in fact it is an isometry if we endow $(L^p(0,1))^m$ with the* *canonical l^p-norm, i.e.,*

$$\|f\|_{X_p} := \left(\sum_{j=1}^m \|f_j\|^p_{L^p(0,1)} \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

or

$$\|f\|_{X_\infty} := \max_{1\le j\le m} \|f_j\|_{L^\infty(0,1)}.$$

In the following we will hence regard X_p as an L^p-space over a finite measure space, so that $X_p \hookrightarrow X_q$ for all $1 \le q \le p \le \infty$, with compact imbedding if $q < p$. Moreover, each X_p is a Banach lattice, and its positive cone can be identified with the positive cone of $L^p(0,m)$.

3. Basic results

We are now in the position to consider an abstract reformulation of our diffusion problem. First we consider the (complex) Hilbert space $X_2 = \left(L^2(0,1)\right)^m$ endowed with the natural inner product

$$(f,g)_{X_2} := \sum_{j=1}^m \int_0^1 f_j(x)\overline{g_j(x)}dx, \qquad f,g \in X_2.$$

On X_2 we define an operator

$$A := \begin{pmatrix} \frac{d}{dx}\left(c_1 \frac{d}{dx}\right) & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m \frac{d}{dx}\right) \end{pmatrix} \tag{3.1}$$

with domain
(3.2)

$$D(A) := \left\{ f \in \left(H^2(0,1)\right)^m : \exists d \in \mathbb{C}^n \text{ s.t. } \begin{array}{c} (\Phi^+)^\top d = f(0),\ (\Phi^-)^\top d = f(1), \\ \text{and } \Phi^+_w f'(0) - \Phi^-_w f'(1) = Bd \end{array} \right\}.$$

We can finally rewrite (NDP) as an abstract Cauchy problem

$$\text{(ACP)} \qquad \left\{ \begin{array}{rcl} \dot u(t) &=& Au(t), \quad t \ge 0, \\ u(0) &=& u_0, \end{array} \right.$$

on X_2. In order to show that A generates a semigroup on X_2 it is convenient to use a variational method.

Lemma 3.1. *Consider the sesquilinear form*

$$\mathfrak{a}(f,g) := \sum_{j=1}^m \int_0^1 c_j(x) f_j'(x)\overline{g_j'(x)}dx - \sum_{i=2}^n b_i f(\mathsf{v}_i)\overline{g(\mathsf{v}_i)}$$

on the Hilbert space X_2 with domain

$$V_0 := \left\{ f \in \left(H^1(0,1)\right)^m : \exists d \in \mathbb{C}^n \textit{ s.t. } (\Phi^+)^\top d = f(0) \textit{ and } (\Phi^-)^\top d = f(1) \right\}.$$

Then $\mathfrak{a}$ is densely defined and enjoys the following properties:

- *(symmetry) : $\mathfrak{a}(f,g) = \overline{\mathfrak{a}(g,f)}$ for all $f,g \in V_0$,*
- *(positivity) : $\mathfrak{a}(f,f) \ge 0$ for all $f \in V_0$,*

- *(closedness) : V_0 is complete for the form norm* $\|f\|_{\mathfrak{a}} := \sqrt{\mathfrak{a}(f,f) + \|f\|^2_{X_2}}$,
- *(continuity) :* $|\mathfrak{a}(f,g)| \leq M\|f\|_{\mathfrak{a}}\|g\|_{\mathfrak{a}}$ *for some $M > 0$ and all $f, g \in V_0$.*

We stress that the sesquilinear form $\mathfrak{a}$ is well-defined, since every $f \in V_0$ is by definition continuous at the vertices $\mathsf{v}_2, \ldots, \mathsf{v}_n$ of the graph, cf. Remark 2.1.

Proof. It is apparent that V_0 is a linear subspace of X_2. Observe that $(C_c^\infty(0,1))^m \subset V_0$. It follows that V_0 is dense in X_2, as by definition $L^2(0,1)$ is the closure of $C_c^\infty(0,1)$ in the L^2-norm. By assumption, the weights c_j are strictly positive while the constants b_i are negative, so that in particular $\mathfrak{a}$ is symmetric and also positive.

In order to check closedness and continuity of $\mathfrak{a}$, observe first that V_0 becomes a Hilbert space whenever equipped with the inner product

$$(f,g)_{V_0} := \sum_{j=1}^m \int_0^1 \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) dx, \qquad f, g \in V_0,$$

since V_0 is a closed subpace of $\left(H^1(0,1)\right)^m$. Further, due to the continuous imbedding of $H^1(0,1)$ into $C[0,1]$, there holds
(3.3)

$$|f(\mathsf{v}_i)| \leq \max_{1\leq j\leq m} \max_{x\in[0,1]} |f_j(x)| \leq \max_{1\leq j\leq m} \|f_j\|_{H^1(0,1)} \leq \sum_{j=1}^m \|f_j\|_{H^1(0,1)} \leq N\|f\|_{V_0},$$

for some constant N and all for $f \in V_0$, $i = 2, \ldots, n$.

Set now

$$c := \min_{1\leq j\leq m} \min_{x\in[0,1]} c_j(x), \quad C := \max_{1\leq j\leq m} \max_{x\in[0,1]} c_j(x), \quad \text{and} \quad b := \sum_{i=2}^n b_i$$

and further

$$\tilde{c} := \min\{c, 1\} \qquad \text{and} \qquad \tilde{C} := \max\{C, -bN^2, 1\}.$$

Then, by (3.3) one has

$$\tilde{c}\|f\|^2_{V_0} \leq \|f\|^2_{\mathfrak{a}} = \sum_{j=1}^m \int_0^1 c_j(x)|f_j'(x)|^2 + |f_j(x)|^2 dx - \sum_{i=2}^n b_i |f(\mathsf{v}_i)|^2 \leq \tilde{C}\|f\|^2_{V_0}, \qquad f \in V_0,$$

Thus, the form norm $\|\cdot\|_{\mathfrak{a}}$ is equivalent to the norm $\|\cdot\|_{V_0}$. Since V_0 is complete with respect to $\|\cdot\|_{V_0}$, the closedness of $\mathfrak{a}$ follows at once.

Finally, $\mathfrak{a}$ is continuous. To see this, take $f, g \in V_0$ and observe that

$$\begin{aligned}
|\mathfrak{a}(f,g)| &\leq C\sum_{j=1}^m \left|\int_0^1 f_j'(x)g_j'(x)dx\right| + \sum_{i=2}^n |b_i||f(\mathsf{v}_i)||g(\mathsf{v}_i)| \\
&\leq C\sum_{j=1}^m \|f_j'\|_{L^2(0,1)}\|g_j'\|_{L^2(0,1)} + \sum_{i=2}^n |b_i|N^2\|f\|_{V_0}\|g\|_{V_0} \\
&\leq \frac{C}{2}\left(\sum_{j=1}^m \|f_j'\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}\left(\sum_{j=1}^m \|g_j'\|^2_{L^2(0,1)}\right)^{\frac{1}{2}} - bN^2\|f\|_{V_0}\|g\|_{V_0} \\
&\leq \left(\frac{C}{2} - bN^2\right)\|f\|_{V_0}\|g\|_{V_0} \leq M\|f\|_{\mathfrak{a}}\|g\|_{\mathfrak{a}}
\end{aligned}$$

by the Cauchy–Schwartz inequality, where $M := \frac{C-2bN^2}{2\tilde{c}}$. □

Lemma 3.2. *The operator associated with the form* $\mathfrak{a}$ *is* $(A, D(A))$ *defined in* (3.1)–(3.2).

Proof. Denote by $(C, D(C))$ the operator associated with $\mathfrak{a}$, which by definition is given by

$$\begin{aligned}
D(C) &:= \{f \in V_0 : \exists g \in X_2 \text{ s.t. } \mathfrak{a}(f,h) = (g,h)_{X_2}\ \forall h \in V_0\}, \\
Cf &:= -g.
\end{aligned}$$

Let us first show that $A \subset C$. Take $f \in D(A)$. Then for all $h \in V_0$

(3.4)
$$\begin{aligned}
\mathfrak{a}(f,h) &= \textstyle\sum_{j=1}^m \int_0^1 c_j(x)f_j'(x)\overline{h_j'(x)}dx - \sum_{i=2}^n b_i f(\mathsf{v}_i)\overline{h(\mathsf{v}_i)} \\
&= \textstyle\sum_{j=1}^m \left[c_j f_j'\overline{h_j}\right]_0^1 - \sum_{j=1}^m \int_0^1 (c_j f_j')'(x)\overline{h_j(x)}dx - \sum_{i=2}^n b_i f(\mathsf{v}_i)\overline{g(\mathsf{v}_i)}.
\end{aligned}$$

Using the incidence matrix $\Phi = \Phi^+ - \Phi^-$ and recalling that a function in V_0 vanishes in the vertex v_1, we can write

$$\sum_{j=1}^m \left[c_j f_j'\overline{h_j}\right]_0^1 = \sum_{j=1}^m\sum_{i=2}^n c_j(\mathsf{v}_i)(\phi_{ij}^- - \phi_{ij}^+)f_j'(\mathsf{v}_i)\overline{h_j(\mathsf{v}_i)}.$$

Due to the continuity of h at the vertices of the graph there exist $d_i := h(v_i) \in \mathbb{C}$ such that $h_j(\mathsf{v}_i) = d_i$ for all $j \in \Gamma(\mathsf{v}_i)$, $i = 1, \ldots, n$. Using the condition $\Phi_w^+ f'(0) - \Phi_w^- f'(1) = Bd$ (which holds for all functions $f \in D(A)$) we obtain that

$$\begin{aligned}
\mathfrak{a}(f,h) &= \sum_{i=2}^n \overline{h(\mathsf{v}_i)}\underbrace{\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+)f_j'(\mathsf{v}_i)}_{=b_i f(\mathsf{v}_i)} - \sum_{j=1}^m \int_0^1 (c_j f_j')'(x)\overline{h_j(x)}dx \\
&\qquad - \sum_{i=2}^n b_i f(\mathsf{v}_i)\overline{h(\mathsf{v}_i)} \\
&= -\sum_{j=1}^m \int_0^1 (c_j f_j')'(x)\overline{h_j(x)}dx = -(Af,h)_{X_2},
\end{aligned}$$

which makes sense because $Af \in X_2$. The proof of the inclusion $A \subset C$ is completed.

To check the converse inclusion $C \subset A$ take $f \in D(C)$. By definition, there exists $g \in X_2$ such that

$$\mathfrak{a}(f,h) = (g,h)_{X_2} = \sum_{j=1}^{m} \int_0^1 g_j(x)\overline{h_j(x)}dx \tag{3.5}$$

for all $h \in V_0$, hence in particular for all functions of the form

$$\begin{pmatrix} 0 \\ \vdots \\ h_j \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ row}, \quad h_j \in H_0^1(0,1).$$

From this follows that (3.5) in fact implies

$$\int_0^1 c_j(x)f_j'(x)\overline{h_j'(x)}dx = \int_0^1 g_j(x)\overline{h_j(x)}dx \text{ for all } j = 1,\ldots,m, \quad h_j \in H_0^1(0,1).$$

By definition of weak derivative this means that $c_j \cdot f_j' \in H^1(0,1)$ for all $j = 1,\ldots,m$. Since $0 < c_j \in C^1[0,1]$, there follows that $f_j' \in H^1(0,1)$ for all $j = 1,\ldots,m$. We conclude that $f \in \left(H^2(0,1)\right)^m$. Moreover, integrating by parts as in (3.4) we see that if (3.5) holds for some $h \in V_0$, then there also necessarily holds

$$\sum_{i=2}^{n} \overline{h(\mathsf{v}_i)} \sum_{j=1}^{m} (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathsf{v}_i) = \sum_{i=2}^{n} b_i f(\mathsf{v}_i)\overline{h(\mathsf{v}_i)}.$$

Since $h \in V_0$ is arbitrary, this means that

$$\sum_{j=1}^{m} (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathsf{v}_i) = b_i f(\mathsf{v}_i) \qquad \text{for all } i = 2,\ldots,n,$$

that is, $\Phi_w^+ f'(0) - \Phi_w^- f'(1) = Bd$. Therefore $f \in D(A)$ and

$$-\sum_{j=1}^{m} \int_0^1 (c_j f_j')'(x)\overline{h_j(x)}dx = \sum_{j=1}^{m} \int_0^1 g_j(x)\overline{h_j(x)}dx$$

holds for all $h \in V_0$. This implies that $Af = -g$, and the proof is complete. □

Corollary 3.3. *The operator $(A, D(A))$ is self-adjoint and strictly negative. Moreover, it has compact resolvent.*

Proof. The self-adjointness and dissipativity of A follow by Lemma 3.1 and 3.2, and [7, Thm. 1.2.1]. Take now $f \in D(A)$ such that $Af = 0$. Then there holds

$$0 = \mathfrak{a}(f,f) = \sum_{j=1}^{m} \int_0^1 c_j(x)|f_j'(x)|^2 dx - \sum_{i=2}^{n} b_i |f(\mathsf{v}_i)|^2 \geq \sum_{j=1}^{m} \int_0^1 c_j(x)|f_j'(x)|^2 dx,$$

where he have used the fact that $b_i \leq 0$, $i = 2,\ldots,n$. Since the weights c_j are strictly positive, this means that f_j is constant for all $j = 1,\ldots,m$. In particular, $f \equiv f(\mathsf{v}_1) = 0$, hence A is one-to-one. Since $D(A) \subset \left(H^2(0,1)\right)^m$, A has compact resolvent and the claim follows. □

Remark 3.4. Taking into account the above corollary, it follows by [3, Thm. 3.14.11] and [9, Lemma 3.1] that A generates a contractive cosine operator function with associated contractive sine operator function. By [2, Cor. 5.6] such cosine and sine operator functions are almost periodic. Moreover, the associated phase space is $V_0 \times X_2$.

Summing up, we conclude that the second order version of (NDP), i.e.,

(NWP)
$$\begin{cases} \ddot{u}_j(t,x) &=& (c_j u_j')'(t,x), & t \geq 0,\ x \in (0,1),\ j=1,\dots,m,\\ u_j(t,\mathsf{v}_1) &=& 0, & t\geq 0,\ j \in \Gamma(\mathsf{v}_1),\\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i), & t \geq 0,\ j,\ell \in \Gamma(\mathsf{v}_i),\ i=2,\dots,n,\\ \sum_{j=1}^m \phi_{ij} c_j(\mathsf{v}_i) u_j'(t,\mathsf{v}_i) &=& b_i u(t,\mathsf{v}_i), & t\geq 0,\ i=2,\dots,n,\\ u_j(0,x) &=& u_{0j}(x), & x \in (0,1),\ j=1,\dots,m,\\ \dot{u}_j(0,x) &=& u_{1j}(x), & x\in(0,1),\ j=1,\dots,m, \end{cases}$$

is well-posed. More precisely, for all initial data $u(0)=u_0 \in V_0$ and $\dot{u}(0)=u_1 \in X_2$ (NWP) admits a unique classical solution that continuously depends on the initial data. Moreover, such a solution satisfies the conservation of energy and it is almost periodic. Such a problem has already been considered in [12, Chapt. 2], where similar results have also been obtained.

By Corollary 3.3 and the spectral theorem the operator A generates a contraction semigroup $(T_2(t))_{t\geq 0}$ on X_2 that is analytic of angle $\frac{\pi}{2}$. This shows that the abstract Cauchy problem (ACP) (and hence the concrete diffusion problem (NDP) on the network) is well-posed in X_2. In fact, much more can be said.

Lemma 3.5. *The semigroup $(T_2(t))_{t\geq 0}$ on X_2, associated with $\mathfrak{a}$, is* sub-Markovian*, i.e., it is real, positive, and contractive on X_∞.*

Proof. By [17, Prop. 2.5, Thm. 2.7, and Cor. 2.17], we need to check that the following criteria are verified for the domain V_0 of $\mathfrak{a}$:

- $f \in V_0 \Rightarrow \overline{f} \in V_0$ and $\mathfrak{a}(\mathrm{Re}f, \mathrm{Im}f) \in \mathbb{R}$,
- $f \in V_0, f$ real-valued $\Rightarrow |f| \in V_0$ and $\mathfrak{a}(|f|,|f|) \leq \mathfrak{a}(f,f)$,
- $0 \leq f \in V_0 \Rightarrow 1 \wedge f \in V_0$ and $\mathfrak{a}(1\wedge f, (f-1)^+) \geq 0$.

It is clear that $\overline{k} \in H^1(0,1)$ if $k \in H^1(0,1)$. Further, if k is real valued, then $|k| \in H^1(0,1)$ and $|k|' = \mathrm{sign}k \cdot k'$, and if $0 \leq k$, then $1 \wedge k \in H^1(0,1)$ with $(1\wedge k)' = k'\mathbb{1}_{\{k<1\}}$ and $((k-1)^+)' = k'\mathbb{1}_{\{k>1\}}$.

By definition, the subspace V_0 contains exactly those functions on the network that are continuous in the vertices and vanish in the vertex v_1. Take any $f \in V_0$. By definition we have $\overline{f_j} = (\overline{f})_j$, $1 \leq j \leq m$. It follows from the above arguments that $\overline{f} \in \left(H^1(0,1)\right)^m$, and one can see that the continuity of the values attained by f in the vertices is preserved after taking the complex conjugate $\overline{f}$. All in all, $\overline{f} \in V_0$. Moreover, $\mathfrak{a}(\mathrm{Re}f, \mathrm{Im}f)$ is the sum of m integrals and $n-1$ numbers. Recall that the weights $c_1, \dots c_m$ are real-valued, positive functions, and that the parameters $b_2, \dots, b_n$ are real, negative numbers. Since all the considered functions are real-valued and all the considered scalars are real, it follows that $\mathfrak{a}(\mathrm{Re}f, \mathrm{Im}g) \in \mathbb{R}$. Thus, the first criterion has been checked.

Moreover, if f is a real-valued function in V_0, then $|f_j| = |f|_j$, $1 \leq j \leq m$, and one sees as above that $|f| \in V_0$. In particular, $||f|'|^2 = |f'|^2$, and there holds

$$\mathfrak{a}(|f|,|f|) = \sum_{j=1}^m \int_0^1 c_j(x)|f_j'(x)|^2 dx - \sum_{i=2}^n b_i |f(\mathsf{v}_i)|^2 = \mathfrak{a}(f,f).$$

This shows that the second criterion applies.

Finally, take $0 \leq f \in V_0$. Then

$$1 \wedge f = 1 \wedge \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 1 \wedge f_1 \\ \vdots \\ 1 \wedge f_m \end{pmatrix},$$

with all the functions $1 \wedge f_j \in H^1(0,1)$ hence $1 \wedge f \in \left(H^1(0,1)\right)^m$. Again, the continuity of function f in the vertices imposes the same property to the function $1 \wedge f$, and hence $1 \wedge f \in V_0$. Further, for all $i = 2, \ldots, n$ there holds

$$(1 \wedge f_j)(\mathsf{v}_i)((f_j - 1)^+)(\mathsf{v}_i) = \begin{cases} f(\mathsf{v}_i) - 1 & \text{if } f(\mathsf{v}_i) \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Summing up,

$$\begin{aligned} \mathfrak{a}(1 \wedge f, (f-1)^+) &= \sum_{j=1}^m \int_0^1 c_j (1 \wedge f_j)'(x)((f_j - 1)^+)'(x) dx \\ &\quad - \sum_{i=2}^n b_i (1 \wedge f_j)(\mathsf{v}_i)((f_j - 1)^+)(\mathsf{v}_i) \\ &= \sum_{j=1}^m \int_0^1 c_j f_j'(x) \mathbb{1}_{\{f_j < 1\}}(x) f_j'(x) \mathbb{1}_{\{f_j > 1\}}(x) dx \\ &\quad - \sum_{\{i : f(\mathsf{v}_i) \geq 1\}} b_i (f(\mathsf{v}_i) - 1) \\ &\geq 0. \end{aligned}$$

We have checked also the third criterion, thus the claim follows. □

Remark 3.6. Let $c, \tilde{c}$ be strictly positive weights of class $\left(C^1[0,1]\right)^m$, such that $c_j(x) \leq \tilde{c}_j(x)$ for all $x \in [0,1]$ and $j = 1, \ldots, m$. Let further $b_2, \ldots, b_n$ and $\tilde{b}_2, \ldots, \tilde{b}_n$ be negative numbers such that $b_i \geq \tilde{b}_i$, $i = 2, \ldots, n$. Denote by $\mathfrak{a}_{c,b}$, $\mathfrak{a}_{\tilde{c},\tilde{b}}$ the form $\mathfrak{a}$ with coefficients c, b and $\tilde{c}, \tilde{b}$, respectively, and by $(T_{c,b}(t))_{t \geq 0}$, $(T_{\tilde{c},\tilde{b}}(t))_{t \geq 0}$ the associated sub-Markovian semigroups. Then the domains of both $\mathfrak{a}_{c,b}$ and $\mathfrak{a}_{\tilde{c},\tilde{b}}$ coincide with V_0. Also, by [17, Prop. 2.20], V_0 is an ideal of itself. A direct computation shows that

$$\mathfrak{a}_{c,b}(f,g) \leq \mathfrak{a}_{\tilde{c},\tilde{b}}(f,g) \qquad \text{for all } 0 \leq f, g \in V_0.$$

It then follows from [17, Thm. 2.24] that $(T_{c,a}(t))_{t \geq 0}$ dominates $(T_{\tilde{c},\tilde{b}}(t))_{t \geq 0}$ in the sense of positive semigroups, i.e.,

$$|T_{\tilde{c},\tilde{b}}(t) f| \leq T_{c,b}(t)|f| \qquad \text{for all } f \in X_2,\ t \geq 0.$$

In other words, if we consider an initial data $u_0 \geq 0$, then the solution to the equation (NDP) with weights $c_1, \ldots c_m$ and parameters $b_2, \ldots, b_n$ attains a.e. a larger value than the solution to (NDP) with weights $\tilde{c}_1, \dots, \tilde{c}_m$ and parameters $\tilde{b}_1, \dots, \tilde{b}_n$.

It has been shown in [10, § 5] that the semigroup governing a diffusion problem on a network *without Dirichlet conditions on any node* is irreducible. In general, this is no more true in the setting considered in this paper.

Definition 3.7. *We call the graph G* almost not connected *if*

- *the set of its edges is the disjoint union of two nonempty subsets $E_1 \dot\cup E_2$ and*
- *if any two edges $e_j \in E_1$ and $e_\ell \in E_2$ are adjacent, then their joint vertex is v_1.*

In other words, G is almost not connected if it can be seen as the union of two non-trivial components that would be no more connected to each other after cutting v_1.

Proposition 3.8. *Let the graph G be almost not connected. Then the semigroup $(T_2(t))_{t\geq 0}$ is not irreducible.*

Proof. By [17, Thm. 2.10] we only need to exhibit a non-trivial subset $\tilde{G}$ of the graph G such that $f\mathbb{1}_{\tilde{G}} \in V_0$ and $\mathrm{Re}\,\mathfrak{a}(f\mathbb{1}_{\tilde{G}}, f\mathbb{1}_{G\setminus\tilde{G}}) \geq 0$ for all $f \in V_0$.

Take then $\tilde{G}$ to be the subset of G consisting of all edges in E_1 and all the adjacent vertices. Let $f \in V_0$. Then $f\mathbb{1}_{\tilde{G}}$ is a function that equals f on the edges of E_1: by definition, $f\mathbb{1}_{\tilde{G}}$ is continuous, it vanishes in v_1 and it satisfies the Kirchhoff-type law at the remaining vertices adjacent to the edges in E_1. Further, $f\mathbb{1}_{\tilde{G}}$ vanishes on the edges of E_2 and all the vertices adjacent to them, thus in particular it is continuous and satisfies the Kirchhoff-type law on them. Summing up, $f\mathbb{1}_{\tilde{G}} \in V_0$ and moreover one sees that $\mathfrak{a}(f\mathbb{1}_{\tilde{G}}, f\mathbb{1}_{G\setminus\tilde{G}}) = 0$. ☐

Remark 3.9. On the other hand, the semigroup may well be irreducible whenever the graph is *not* almost not connected. This can be seen by considering, e.g., a diffusion problem on a network represented by a triangle with vertices $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$ with Dirichlet condition on v_1 and Kirchhoff-type law on $\mathsf{v}_2, \mathsf{v}_3$.

4. Ultracontractivity and Gaussian estimates

Lemma 4.1. *The semigroup $(T_2(t))_{t\geq 0}$ on X_2 associated with $\mathfrak{a}$ is ultracontractive. In particular, it satisfies the estimate*

$$\|T_2(t)f\|_{X_\infty} \leq Mt^{-\frac{1}{4}}\|f\|_{X_2} \qquad \text{for all } t > 0,\ f \in X_2, \tag{4.1}$$

for some constant M.

Proof. By [17, Thm. 6.3] it suffices to show that there holds

$$\|f\|_{X_2} \leq M\mathfrak{a}(f,f)^{\frac{1}{6}} \cdot \|f\|_{X_1}^{\frac{2}{3}} \qquad \text{for all } f \in V_0,$$

for some constant M. Recall that

$$\begin{aligned}\|k\|_{L^2(0,1)} &\leq M_1 \left(\|k'\|_{L^2(0,1)} + \|k\|_{L^1(0,1)}\right)^{\frac{1}{3}} \cdot \|k\|_{L^1(0,1)}^{\frac{2}{3}} \\ &\leq M_1 \|k\|_{H^1(0,1)}^{\frac{1}{3}} \cdot \|k\|_{L^1(0,1)}^{\frac{2}{3}},\end{aligned}$$

is valid for all $k \in H^1(0,1)$ and some constant M_1, cf. [15, Thm. 1.4.8.1].

Take finally $f \in V_0$ and observe that by the above Nash-type inequality

$$\begin{aligned}\|f\|_{X_2}^2 &= \sum_{j=1}^m \|f_j\|^2_{L^2(0,1)} \le M_1 \sum_{j=1}^m \|f_j\|^{\frac{2}{3}}_{H^1(0,1)} \cdot \|f_j\|^{\frac{4}{3}}_{L_1(0,1)} \\ &\le M_2 \left(\sum_{j=1}^m \|f_j\|_{H^1(0,1)}\right)^{\frac{2}{3}} \cdot \left(\sum_{j=1}^m \|f\|_{L^1(0,1)}\right)^{\frac{4}{3}} \\ &\le M_3 \|f\|^{\frac{2}{3}}_{V_0} \cdot \|f\|^{\frac{4}{3}}_{X_1}.\end{aligned}$$

Finally, observe that, since by Lemma 3.3 the operator A associated with $\mathfrak{a}$ is self-adjoint and strictly negative,

$$|||f||| := \sqrt{\mathfrak{a}(f,f)}, \qquad f \in V_0$$

defines an equivalent norm on V_0 and the claim follows. □

Remark 4.2. In the terminology of N.Th. Varopoulos ([19, § 0.1], cf. also [1, § 7.3.2]), Lemma 4.1 says that the dimension of the semigroup $(T_2(t))_{t\ge 0}$ is 1. This is true regardless of the structure of the underlying graph.

The following now holds by [7, Thm. 1.4.1, Thm. 1.6.4, and Thm. 2.1.5].

Corollary 4.3. *The semigroup $(T_2(t))_{t\ge 0}$ extends to a family of compact, contractive, real, positive semigroups $(T_p(t))_{t\ge 0}$ on X_p, $p \in [1,\infty]$. Such semigroups are strongly continuous if $p \in [1,\infty]$.*

Moreover, the spectrum of A_p is independent of p, where A_p denotes the generator of $(T_p(t))_{t\ge 0}$. All the eigenfunctions of $A = A_2$ are of class X_∞.

In [10, § 5] it is shown that the solution to a diffusion problem on a network converges toward an equilbrium at a pace that depends on the largest nonnegative eigenvalue of the diffusion operator. Likewise, in our context we can state the following.

Proposition 4.4. *All the semigroups $(T_p(t))_{t\ge 0}$, $p \in [1,\infty]$, are uniformly exponentially stable, their common growth bound being given by the strictly negative spectral bound $s(A)$ of the operator A.*

Proof. It follows from Corollary 3.3 that $s(A) < 0$. Define the perturbed form

$$\tilde{\mathfrak{a}}(f,g) := \mathfrak{a}(f,g) - s(A)(f,g)_{X_2}, \qquad f,g \in V_0.$$

Then $\tilde{\mathfrak{a}}$ is sesquilinear, densely defined, symmetric, closed, and positive. It follows as in Corollary 4.3 that the associated semigroups are contractive on all X_p, i.e., there holds

$$\|T_p(t)f\|_{X_p} \le e^{ts(A)}\|f\|_{X_p} \qquad \text{for all } t \ge 0,\ f \in X_p,$$

for all $p \in [1,\infty]$. □

Remark 4.5. Consider the case of $c_1 = \ldots = c_m \equiv 1$ and $b_2 = \ldots = b_n = 0$. Then, it follows by [16, Théo. 2.4 and Théo. 3.1] that

$$-\left(\frac{\pi}{m+1}\right)^2 \le s(A) \le -\left(\frac{\pi}{2m}\right)^2.$$

In other words, taking into account Proposition 4.4 we can say that *the more edges belong to the network, the slower is the heat dissipation.*

A further, more involved upper estimate on $s(A)$ is shown in [16, Théo. 3.2], showing that the inner structure of the graph does influence the asymptotical behavior of the diffusion problem.

Combining the above uniform exponential stability and ultracontractivity results we can finally show that $X_2 - X_\infty$ uniform exponential stability holds.

Corollary 4.6. *The semigroup $(T_2(t))_{t\geq 0}$ on X_2 satisfies the estimate*

$$\|T_2(t)f\|_{X_\infty} \leq M \left(\frac{1-ts(A)}{t}\right)^{\frac{1}{4}} e^{ts(A)}\|f\|_{X_2} \qquad \text{for all } t>0,\ f\in X_2,$$

where M is the constant that appears in (4.1)

Proof. Taking into account Lemma 4.1 and Proposition 4.4, the claim follows directly from [17, Lemma 6.5]. □

Remarks 4.7. 1) As a direct consequence of the ultracontractivity of $(T_2(t))_{t\geq 0}$ and the Dunford-Petty criterion, the semigroup has an integral kernel for all $t>0$, cf. [7, Lemma 2.1.2]. More precisely, denote by $(\tilde{T}_p(t))_{t\geq 0}$ the semigroup on $L^p(0,m)$ that is similar to $(T_p(t))_{t\geq 0}$ on X_p with a similarity transformation given by the isometry U introduced in Definition 2.3. Then for all $p\in[1,\infty]$ the action of $(\tilde{T}_p(t))_{t\geq 0}$ is given by

$$\tilde{T}_p(t)g(\cdot) = \int_0^m K_t(\cdot,y)g(y)dy, \qquad t>0,\ g\in L^p(0,m),$$

for a suitable kernel $K_t \in L^\infty\big((0,m)\times(0,m)\big)$. Further, the uniform bound

$$0\leq K_t(x,y)\leq M^2\sqrt{\frac{2}{t}} \qquad \text{for all } t>0, \quad \text{a.e. } x,y\in(0,m),$$

holds, where M is the constant that appears in (4.1).

2) We can reformulate Lemma 4.1 and say that $(T_2(t))_{t\geq 0}$ satisfies the estimate

$$\|T_2(t)f\|_{X_\infty} \leq e^{\kappa(t)}\|f\|_{X_2} \qquad \text{for all } t>0,\ f\in X_2,$$

where

$$\kappa(t) := \log M - \frac{1}{4}\log t. \tag{4.2}$$

A direct computation shows that κ is a continuous, monotonically decreasing function on $(0,\infty)$. We thus apply [7, Thm. 2.2.3] and obtain the logarithmic Sobolev inequality

$$\int_0^m \tilde{f}^2 \log \tilde{f} dx \leq \varepsilon\mathfrak{a}(f,f) + \kappa(\varepsilon)\|f\|^2_{X_2} + \|f\|^2_{X_2}\log\|f\|_{X_2}, \tag{4.3}$$

which is valid for all $0\leq f\in V_0$ and all $\epsilon>0$ (here $L^2(0,m)\ni\tilde{f}=Uf$ denotes the function isometric to f as in Definition 2.3).

Such a logarithmic Sobolev inequality is extremely powerful. In particular, it can be used to prove Gaussian estimates for our semigroup, following the methods developed by Davies (see [7, § 3.2] and references therein).

Theorem 4.8. *The Gaussian upper bound*

$$0 \le K_t(x,y) \le c_\delta t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4(1+\delta)Ct}} \qquad \text{for all } t>0,\ \delta\in(0,1),\ x,y\in(0,m), \tag{4.4}$$

holds for the kernel K_t*, for some constant* $c_\delta > 0$*, where* $C := \max_{x\in[0,m]} \tilde{c}(x)$*,* $\tilde{c}$ *being as in Definition 2.3.*

Proof. One sees that $\tilde{c} \in C^1((0,1)\cup\ldots\cup(m-1,m))$ and that there exist constants c, C such that $0 < c \le \tilde{c}(x) \le C$ for a.e. $x \in (0,m)$. Set

$$\Psi := C^\infty((0,m);\mathbb{R}) \cap C_b((0,m);\mathbb{R}),$$

and define a metric on $(0,m)$ by

$$d(x,y) := \sup\left\{|\psi(x)-\psi(y)| : \psi\in\Psi, \tilde{c}(x)|\psi'(x)|^2 \le 1 \text{ for a.e. } x\in(0,m)\right\}.$$

Then, taking into account (4.2) and (4.3), by [7, Thm. 3.2.7] the kernel K_t satisfies

$$0 \le K_t(x,y) \le c_\delta t^{-\frac{1}{2}} e^{-\frac{d(x,y)^2}{4(1+\delta)t}} \qquad \text{for all } t>0,\ \delta\in(0,1),\ x,y\in(0,m),$$

for some constant $c_\delta > 0$. However, observe that

$$d(x,y) \ge \sup\{|\psi(x)-\psi(y)| : \sqrt{C}|\psi'(x)|^2 \le 1 \text{ for a.e. } x\in(0,m)\} = \frac{|x-y|}{\sqrt{C}}.$$

The claim now follows. □

We are finally able to obtain an optimal result on the analiticity of the semigroup generated by A.

Corollary 4.9. *All the semigroups* $(T_p(t))_{t\ge0}$*,* $p\in[1,\infty)$*, are analytic of angle* $\frac{\pi}{2}$*.*

Proof. Recall that A is self-adjoint and dissipative, hence $(T_2(t))_{t\ge0}$ is analytic of anglee $\frac{\pi}{2}$. Then the claim follows from Theorem 4.8 and [17, Thm. 6.16]. □

Remark 4.10. As shown among others by Arendt, Duong, ter Elst, Ouhabaz, Robinson, Gaussian estimates like (4.4) are a key argument for discussing a number of different issues that go far beyond the scope of this paper. Without going into details, we recall that Theorem 4.8 implies at once, among other, the property of maximal regularity for $(T_p(t))_{t\ge0}$ for $p\in(1,\infty)$, upper estimates for the time derivative of the heat kernel K_t, L^p-estimates for Schrödinger and wave equations, and the fact that A_p has bounded H^∞-calculus on each sector (and therefore that it has bounded imaginary powers) for $p\in(1,\infty)$, cf. [17, § 6.5 and Chapt. 7], [1, § 7.4] and references therein.

In order to discuss the well-posedness of the problem in an L^p-setting, we want to identify the generators of the semigroups $(T_p(t))_{t\ge0}$.

Lemma 4.11. *For all* $p\in(1,\infty]$ *the generator* A_p *of the semigroup* $(T_p(t))_{t\ge0}$ *is given by the operator whose action on the domain*

$$D(A_p) := \left\{ f \in \left(W^{2,p}(0,1)\right)^m : \exists d\in\mathbb{C}^n \text{ s.t. } \begin{array}{c} (\Phi^+)^\top d = f(0),\ (\Phi^-)^\top d = f(1), \\ \text{and } \Phi_w^+ f'(0) - \Phi_w^- f'(1) = Bd \end{array} \right\}.$$

is formally given in (3.1).

Proof. We only need to prove the claim for $p > 2$, the general case then following by duality. We have already remarked that $X_p \hookrightarrow X_q$ for all $1 \leq q \leq p \leq \infty$. Moreover, it follows from the ultracontractivity of $(T_2(t))_{t\geq 0}$ (see Remark 4.7.(2)) that X_p is invariant under $(T_p(t))_{t\geq 0}$ for all $p > 2$. Thus, by [8, Prop. II.2.3] the generator of $(T_p(t))_{t\geq 0}$ is the part of A in X_p. A direct computations yields the claim. □

Theorem 4.12. *The first order problem* (NDP) *is well-posed on X_p, $p \in (1,\infty)$, i.e., for all initial data $u_0 \in X_p$ the problem* (NDP) *admits a unique classical solution that continuously depends on the initial data.*

Such a solution essentially bounded in the space variable and its ∞-norm tends to 0 in time. If further $c_j \in C^\infty[0,1]$, $j = 1,\dots,m$, then the solution $u(t,\cdot)$ is of class C^∞ with respect to the space variable.

Proof. The well-posedness and boundedness results follow from the fact that the semigroup $(T_2(t))_{t\geq 0}$ is ultracontractive and extends to a family of semigroups $(T_2(p))_{t\geq 0}$ that, by Lemma 4.11, actually govern (NDP). The decay of the solution is ensured by the uniform exponential stability of all semigroups.

Finally, if $c_j \in C^\infty[0,1]$, $j = 1,\dots,m$, then one sees that $D(A_p^\infty) \subset (C^\infty[0,1])^m$ for all $p \in (1,\infty)$. Since the semigroup $(T_p(t))_{t\geq 0}$ is analytic, it maps X_p into $D(A_p^\infty)$, and the claim follows. □

Observe that if in (NDP) we replace the Dirichlet condition in v_1 by continuity of the values of $u_j(t,\mathsf{v}_1)$, $t \geq 0$, $j \in \Gamma(\mathsf{v}_1)$, plus a Kirchhoff-type condition analogous to that imposed on the other nodes, we obtain the system

(4.5)
$$\left\{\begin{array}{rcll} \dot{u}_j(t,x) &=& (c_j u_j')'(t,x), & t \geq 0,\ x \in (0,1),\ j = 1,\dots,m,\\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i), & t \geq 0,\ j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n,\\ \sum_{j=1}^m \phi_{ij} c_j(\mathsf{v}_i) u_j'(t,\mathsf{v}_i) &=& b_i u(t,\mathsf{v}_i), & t \geq 0,\ i = 1,\dots,n,\\ u_j(0,x) &=& u_{0j}(x), & x \in (0,1),\ j = 1,\dots,m, \end{array}\right.$$

where $b_2,\dots,b_n$ are the same parameters appearing in (NDP) and b_1 is an arbitrary negative number. Such an initial-value problem has been proven to be well-posed in [10][2]: we can compare its solution and that to (NDP) and obtain the following.

Proposition 4.13. *The semigroup $(T_2(t))_{t\geq 0}$ governing* (NDP) *is dominated by the semigroup $(\tilde{T}_2(t))_{t\geq 0}$ governing* (4.5) *in the sense of positive semigroups.*

Proof. As shown in [10], $(\tilde{T}_2(t))_{t\geq 0}$ is a sub-Markovian semigroup that comes from a form with domain

$$V = \left\{ f \in \left(H^1(0,1)\right)^m : \exists d \in \mathbb{C}^n \text{ s.t. } (\tilde{\Phi}^+)^\top d = f(0) \text{ and } (\tilde{\Phi}^-)^\top d = f(1) \right\},$$

where $\tilde{\Phi}^+ = (\tilde{\phi}^+_{ij})$ and $\tilde{\Phi}^- = (\tilde{\phi}^-_{ij})$ represent the incindence matrices defined by

$$\tilde{\phi}^+_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(0) = \mathsf{v}_i,\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad \tilde{\phi}^-_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(1) = \mathsf{v}_i,\\ 0, & \text{otherwise.}\end{cases}$$

Since a Dirichlet condition in the node v_1 implies in particular continuity on a function in that vertex, one sees that $V_0 \subset V$. Accordingly, by [17, Cor. 2.22] it

[2]In fact, only the special case of $b_i = 0$, $i = 1,\dots,n$, has been considered in [10], but the results there can be shown to prevail also in the more general setting.

suffices to prove that V_0 is an ideal of V, i.e., that the following conditions are satisfied:

- $f \in V_0 \Rightarrow |f| \in V$,
- $f \in V_0,\ g \in V,$ and $|g| \leq |f| \Rightarrow g \cdot \mathrm{sign} f \in V_0$.

To check the first condition, observe that $H^1_0(0,1)$ is an ideal of $H^1(0,1)$, and that the continuity of the values of $f \in \left(H^1(0,1)\right)^m$ in the nodes is not affected by taking the absolute value of f. Let now $f \in V_0$ and $g \in V$. If $|g| \leq |f|$, then in particular $|g_j(\mathsf{v}_1)| \leq |f_j(\mathsf{v}_1)| = 0$ for all $j \in \Gamma(\mathsf{v}_1)$, i.e., $g \in V_0$. Finally, since A generates a positive semigroup, V_0 is an ideal of itself and this yields that $g \cdot \mathrm{sign} f \in V_0$. □

5. The heat equation on spaces of continuous functions

Consider now the part $\tilde{A}_\infty$ of A in the Banach space $\tilde{X}_\infty := (C[0,1])^m$, whose domain is given by

$$D(\tilde{A}_\infty) = \left\{ f \in \left(C^2(0,1) \cap C^1[0,1]\right)^m : \exists d \in \mathbb{C}^n \text{ s.t. } \begin{array}{c} (\Phi^+)^\top d = f(0),\ (\Phi^-)^\top d = f(1), \\ \text{and } \Phi^+_w f'(0) - \Phi^-_w f'(1) = Bd \end{array} \right\}.$$

The main motivation for considering semigroups on $(C[0,1])^m$ comes from applications involving semilinear equations, since we can then effectively apply the theory developed, e.g., in [13, Chapt. 7]. As an elementary, yet motivating example we mention the following.

Proposition 5.1. *Consider functions $\phi_j \in C^2(\mathbb{R})$, $j = 1, \dots, m$. Then for all $u_0 \in (C[0,1])^m$ the semilinear problem*

(5.1)
$$\begin{cases} \dot{u}_j(t,x) &=& (c_j u_j')'(t,x) & \\ & & \quad +(\phi_j(u_j(t,x))', & t>0,\ x \in (0,1),\ j = 1,\dots,m, \\ u_j(t,\mathsf{v}_1) &=& 0, & t>0,\ j \in \Gamma(\mathsf{v}_1), \\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i), & t>0,\ j,\ell \in \Gamma(\mathsf{v}_i),\ i = 2,\dots,n, \\ \sum_{j=1}^m \phi_{ij} c_j(\mathsf{v}_i) u_j'(t,\mathsf{v}_i) &=& b_i u(t,\mathsf{v}_i), & t>0,\ i = 2,\dots,n, \\ u_j(0,x) &=& u_{0j}(x), & x \in (0,1),\ j = 1,\dots,m, \end{cases}$$

admits a unique (global) mild solution u that depends continuously (with respect to the sup-norm) on the initial data. In fact, u satisfies the problem pointwise for $t > 0$.

Proof. Rewrite (5.1) as a semilinear asbtract Cauchy problem

$$\begin{cases} \dot{u}(t) &=& \tilde{A}_\infty u(t) + \Phi(u(t)), \quad t>0, \\ u(0) &=& u_0, \end{cases}$$

on $\tilde{X}_\infty$. Here Φ is the nonlinear operator defined by

$$\Phi(u)(\cdot) := \begin{pmatrix} \frac{d}{dx}\left(\phi_1(u_1(\cdot))\right) \\ \vdots \\ \frac{d}{dx}\left(\phi_m(u_m(\cdot))\right) \end{pmatrix}.$$

By Corollary 4.3 and Corollary 4.9 we deduce that all the operators A_p, $p \in [1,\infty]$, are dissipative and sectorial of angle $\frac{\pi}{2}$. In particular, for each $\epsilon \in (0, \frac{\pi}{2})$ there exists $M_\epsilon \geq 1$ such that the estimate

$$\|\lambda R(\lambda, A_\infty)\|_{\mathcal{L}(X_\infty)} \leq M_\epsilon \tag{5.2}$$

holds for all $\lambda \in \{\mu \in \mathbb{C} : |\arg \mu| < \pi - \epsilon\}$.

Now observe that if $f \in \tilde{X}_\infty$, then $R(\lambda, A_\infty)f = R(\lambda, A_2)f \in D(A_2^\infty)$. But one has $D(A_2) \subset \left(H^2(0,1)\right)^m$, so that

$$D(A_2^\infty) \subset \left(\bigcap_{k=1}^{\infty} H^k(0,1)\right)^m = \left(C^\infty[0,1]\right)^m$$

and $R(\lambda, A_\infty)f$ is a continuous function for all $\lambda \in \{\mu \in \mathbb{C} : |\arg \mu| < \pi - \epsilon\}$. It follows that the analogous of (5.2) also holds with respect to the norm of $\tilde{X}_\infty$, hence $\tilde{A}_\infty$ is sectorial, and we are in a setting that is analogous to that of [13, § 7.3.3]. Now, mimicking the proof of [13, Prop. 7.3.6] the claim follows. □

A thorough treatment of well-posedness and stability semilinear diffusion problems over networks goes beyond the scope of this paper. We will deal with such an issue in a forthcoming paper.

Even in the linear case (i.e., $\phi_1 \equiv 0$, $j = 1, \ldots, m$), the problem (5.1) is not well-posed in a classical sense. In fact, albeit sectorial (hence the generator of an analytic semigroup), the operator $\tilde{A}_\infty$ is not densely defined in $\tilde{X}_\infty$, thus the generated semigroup is not strongly continuous.

Let us define

$$C_0(G) := \left\{ f \in (C[0,1])^m : \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \right\},$$

which by the theorem of Stone–Weierstrass is the closure of $D(\tilde{A}_\infty)$. (Observe that $C_0(G)$ can be looked at as the space of all continuous functions on the graph G that vanish in v_1). The following generation result is the main result of this paper.

Theorem 5.2. *The part* $\mathbf{A}$ *of* $\tilde{A}_\infty$ *in* $C_0(G)$ *generates a compact, contractive, real, positive, strongly continuous semigroup. Such a semigroup is analytic of angle* $\frac{\pi}{2}$ *and uniformly exponentially stable.*

Proof. Reasoning as in the proof of Proposition 5.1, we deduce from Corollary 4.3 that $\mathbf{A}$ is a resolvent positive operator on $C_0(G)$. Since $\mathbf{A}$ is also densely defined, by [3, Thm. 3.11.9] it generates a positive strongly continuous semigroup $(\mathbf{T}(t))_{t\geq 0}$.

Again as in the proof of Proposition 5.1, we see that $\mathbf{A}$ is sectorial and dissipative: this yields the analyticity (with angle $\frac{\pi}{2}$) and the contractivity of $(\mathbf{T}(t))_{t\geq 0}$. Observe further that the p-independence of the spectrum of A_p (by Corollary 4.3) yields the invertibility of $\mathbf{A}$, hence the uniform exponential stability of $(\mathbf{T}(t))_{t\geq 0}$.

Finally, in order to show that the semigroup is compact, observe that due to its analyticity $T_2(t)$ maps X_2 into $D(A_2^\infty) \subset (C^\infty[0,1])^m \cap C_0(G) \subset D(\mathbf{A})$ for all $t > 0$. Thus, denoting by $X_\mathbf{A}$ the Banach space obtained by endowing $D(\mathbf{A})$ with the graph norm, we have

$$\mathbf{T}(t) = i_{X_\mathbf{A}, C_0(G)} \circ T_2(t) \circ i_{C_0(G), X_2}, \qquad t > 0.$$

Here $i_{C_0(G),X_2}$ and $i_{X_\mathbf{A},C_0(G)}$ denote the canonical imbeddings of $C_0(G)$ into X_2 and of $X_\mathbf{A}$ into $C_0(G)$, respectively. It follows from the theorem of Ascoli-Arzelà that the latter imbedding is compact, so that also $\mathbf{T}(t)$ is compact for $t > 0$, and the claim follows. □

We can finally draw a conclusion that is similar to Theorem 4.12, and can be proven likewise.

Theorem 5.3. *The first order problem* (NDP) *is well-posed on $C_0(G)$, i.e., for all initial data $u_0 \in C_0(G)$ the problem* (NDP) *admits a unique classical solution that continuously depends on the initial data. The sup-norm of the solution tends to 0 in time.*

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